A METHOD OF STUDYING THE STATE OF A RAREFIED GAS

A. N. Temchin

UDC 533.7

We propose a mathematically correct method of solving the Boltzmann equations.

The evolution of the state of a neutral or weakly ionized gas can be described [1] by means of the familiar kinetic equation

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}+\frac{\mathbf{X}}{m}\frac{\partial}{\partial \mathbf{v}}\right)f=\int \sigma\,|\mathbf{v}-\underset{*}{\mathbf{v}}|\,(f'\underset{*}{f'}-f\underset{*}{f})\sin\vartheta\, d\vartheta\, d\varphi\, d\underset{*}{\mathbf{v}}. \tag{1}$$

Here $f\overset{\text{def}}{=}f(t,\mathbf{r},\mathbf{v})$ and $\underset{*}{f}\overset{\text{def}}{=}f(t,\mathbf{r},\underset{*}{\mathbf{v}})$ are distribution functions for the particles prior to collision while $f'\overset{\text{def}}{=}f(t,\mathbf{r},\mathbf{v}')$ and $\underset{*}{f'}\overset{\text{def}}{=}f(t,\mathbf{r},\underset{*}{\mathbf{v}'})$ are distribution functions subsequent to collision; $\sigma\overset{\text{def}}{=}\sigma(|\mathbf{v}-\underset{*}{\mathbf{v}}|,\vartheta,\varphi)$ is the differential scattering cross section for the angles ϑ and φ in the center-of-mass system formed by the colliding molecules; **X** is the external field; m is the particle mass; the integration is carried out over all values of the angles and of the components of the velocity $\underset{*}{\mathbf{v}}$.

The most fully developed methods of solving (1) are based either on the series expansion of $f(t,\mathbf{r},\mathbf{v})$ and of the operator $\partial/\partial t$ in terms of the small parameter (Enskog–Chapman, [2]), or on the series expansion of $f(t,\mathbf{r},\mathbf{v})$ on the basis of a specially constructed system of orthonormal functions (Grad, [2]). In either case, the calculation of each successive approximation involves ever-increasing difficulties. Moreover, the Enskog series converges only asymptotically, as $\mathrm{Kn}\to 0$, while the Grad series converges only on the average, and only if $f(t,\mathbf{r},\mathbf{v})$ diminishes more rapidly as $|\mathbf{v}|\to\infty$ then $\exp\{-|\mathbf{v}|^2/4\}$; we do not know whether the unknown function exhibits this property at any instant of time.

Other methods of constructing the approximate solutions involve the direct proof of the theorems of existence, which are all the more important, since there is presently no fault-free method of deriving (1). A number of references ([3]-[9]) are devoted to the problem of whether or not solutions exist for the Boltzmann equation. Since the properties of the collision integral are substantially different for the various potentials of particle interaction, we most frequently examine a gas of Maxwell or pseudo-Maxwell molecules ([4]-[7]).

Below we prove the local theorem on the existence and uniqueness of the solution for Eq. (1), with rather broad assumptions as to the potential of particle interaction, as well as the uniform convergence of the sequence of approximations of the exact solution.

The Equation. Let us assume a gas of classical point particles which interact with each other in the following manner.

We assume that the potential $\varphi(r_{ij})$ $(i\neq j)$ of the molecular field is centrally symmetrical and, moreover, that

$$\varphi(r_{ij})\begin{cases}\sim 1/r_{ij}^{\nu-1}, & r_{ij}\leqslant r_0,\\ \equiv 0, & r_{ij}>r_0,\end{cases}\qquad r_0=\text{const}>0. \tag{2}$$

The collision integral can then be written as

$$If=\int K(f'\underset{*}{f'}-f\underset{*}{f})\,d\omega\, d\underset{*}{\mathbf{v}}, \tag{3}$$

A. V. Topchiev Institute of Petrochemical Synthesis, Academy of Sciences of the USSR, Moscow.
Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 16, No. 5, pp. 885-891, May, 1969. Original article submitted July 11, 1968.

where

$$K \overset{\text{def}}{=} |\mathbf{v} - \underset{*}{\mathbf{v}}|^{\gamma} K(\vartheta), \quad d\omega \overset{\text{def}}{=} d\vartheta d\varphi, \quad \gamma = \frac{\nu - 5}{\nu - 1},$$

in which case

$$\int K(\vartheta)\, d\omega < \infty. \tag{4}$$

For models of solid spheres with a diameter d we have

$$K = d^2 |\mathbf{v} - \underset{*}{\mathbf{v}}| \sin\vartheta \cos\vartheta. \tag{5}$$

Let us assume that the gas occupies the entire space and that there are no external fields. With these limitations, the Boltzmann equation has the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right) f = \int K(f'\underset{*}{f'} - f\underset{*}{f})\, d\omega d\underset{*}{\mathbf{v}}. \tag{6}$$

Since the particle masses are identical, the following relationships are valid:

$$\begin{gathered} |\mathbf{v}|^2 + |\underset{*}{\mathbf{v}}|^2 = |\mathbf{v}'|^2 + |\underset{*}{\mathbf{v}'}|^2, \\ \mathbf{v}' = \mathbf{v} + \mathbf{n}(\mathbf{n}, \underset{*}{\mathbf{v}} - \mathbf{v}), \quad \underset{*}{\mathbf{v}'} = \underset{*}{\mathbf{v}} - \mathbf{n}(\mathbf{n}, \underset{*}{\mathbf{v}} - \mathbf{v}), \\ \mathbf{n} = \{\cos\vartheta, \ \sin\vartheta\cos\varphi, \ \sin\vartheta\sin\varphi\}. \end{gathered} \tag{7}$$

We will solve the Cauchy problem

$$f(0, \mathbf{r}, \mathbf{v}) = f_0(\mathbf{r}, \mathbf{v}). \tag{8}$$

The Solution. Let us introduce the numerical parameters α, χ, and χ_1 such that

$$\begin{gathered} \alpha, \ \chi, \ \chi_1 > 0, \quad \chi > \chi_1, \\ [\alpha] = [\mathbf{v}]^{-2}[t]^{-1}, \ [\chi] = [\chi_1] = [t]. \end{gathered} \tag{9}$$

The solution of the problem for the times $t \in (0, \chi_1)$ will be sought in the form

$$f(t, \mathbf{r}, \mathbf{v}) \overset{\text{def}}{=} F(t, \mathbf{r}, \mathbf{v}) \exp\{-\alpha(\chi - t)|\mathbf{v}|^2\}, \tag{10}$$

where $F(t, \mathbf{r}, \mathbf{v})$ is a new unknown function. Assuming $f_0(\mathbf{r}, \mathbf{v})$ to be limited and diminishing rather rapidly as $|\mathbf{v}| \to \infty$ we require that $F_0(\mathbf{r}, \mathbf{v}) \overset{\text{def}}{=} f_0(\mathbf{r}, \mathbf{v})\exp\{\alpha\chi|\mathbf{v}|^2\}$ be limited. We then find the condition for the numerical value of the product $\alpha\chi$, i.e.,

$$f_0(\mathbf{r}, \mathbf{v}) \exp\{\alpha\chi|\mathbf{v}|^2\} < \infty. \tag{11}$$

The function $F \overset{\text{def}}{=} F(t, \mathbf{r}, \mathbf{v})$ satisfies the equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \alpha|\mathbf{v}|^2\right) F = \int K \exp\{-\alpha(\chi - t)|\underset{*}{\mathbf{v}}|^2\}(F'\underset{*}{F'} - F\underset{*}{F})\, d\omega d\underset{*}{\mathbf{v}} \tag{12}$$

with the initial equation

$$F(0, \mathbf{r}, \mathbf{v}) = F_0(\mathbf{r}, \mathbf{v}) \overset{\text{def}}{=} F_0. \tag{13}$$

The following integral equation corresponds to this problem:

$$\begin{gathered} F = \int_0^t \exp\{-\alpha|\mathbf{v}|^2(t-\tau)\}\left[\int K_e(F'\underset{*}{F'} - F\underset{*}{F})\, d\omega d\underset{*}{\mathbf{v}}\right]_{t-\tau} d\tau + [F_0]_t \exp\{-\alpha|\mathbf{v}|^2 t\}, \\ K_e \overset{\text{def}}{=} K \exp\{-\alpha(\chi - t)|\underset{*}{\mathbf{v}}|^2\}, \end{gathered} \tag{14}$$

or in operator notation

$$F = AF, \quad A \overset{\text{def}}{=} A_1 + [F_0]_t \exp\{-\alpha|\mathbf{v}|^2 t\}. \tag{15}$$

In Eq. (14) the brackets $[\ ]_s$ denote a "backward" shift operator (by time s) with respect to the characteristics of the equation:

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)u=0.$$

As is usual, we will use C to denote the space of continuous bounded functions which depend on the variables t, $\mathbf{r}$, and $\mathbf{v}$, with $0 \le t \le \chi_1$, $|\mathbf{r}| \le \infty$, $|\mathbf{v}| \le \infty$. The distance function in C is determined from the expression

$$\|x-y\|=\max_{t,\mathbf{r},\mathbf{v}}|x-y|, \quad x,\ y\in C.$$

We will demonstrate that if $F_0 \in C$, we have $AC \subset C$. Indeed, in this case

$$[F_0]_t \exp\{-\alpha|\mathbf{v}|^2 t\}\in C,$$

and with $x \in C$

$$\|A_1x\|\leqslant 2\|x\|^2\left\|\int_0^t \exp\{-\alpha|\mathbf{v}|^2(t-\tau)\}\left[\int K_e d\omega d\underset{*}{\mathbf{v}}\right]_{t-\tau}d\tau\right\|. \tag{16}$$

Let us introduce the function

$$L(t,\ \mathbf{v})\overset{\text{def}}{=}\int_0^t \exp\{-\alpha|\mathbf{v}|^2(t-\tau)\}\left[\int K_e d\omega d\underset{*}{\mathbf{v}}\right]_{t-\tau}d\tau. \tag{17}$$

Since K_e increases as $|\mathbf{v}-\underset{*}{\mathbf{v}}| \to \infty$ no more rapidly than linearly, L(t, $\mathbf{v}$) is limited and exhibits the following property:

$$L(t,\ \mathbf{v})\underset{t\to 0}{\to}0. \tag{18}$$

For example, it is easy to show that for the solid-sphere model

$$L(t,\ \mathbf{v})\leqslant 2S(|\mathbf{v}|I_1+I_2)\frac{1-\exp\{-\alpha|\mathbf{v}|^2t\}}{\alpha|\mathbf{v}|^2}<\infty, \tag{19}$$

where S is the complete scattering cross section, and I_1 and I_2 are certain integrals which can be calculated in elementary functions, i.e.,

$$I_1=\left(\frac{\pi}{\alpha(\chi-\chi_1)}\right)^{3/2},\qquad I_2=\frac{\pi^{3/2}}{\Gamma(3/2)\left(\alpha(\chi-\chi_1)\right)^2}. \tag{20}$$

The operator A thus is effective in the space C.

Let A be given in the elements of a closed sphere $S(\vartheta, R) \subset C$; $\vartheta \overset{\text{def}}{=} \vartheta(t, \mathbf{r}, \mathbf{v}) \equiv 0$ is the null of the space C. Let us determine under which conditions A is a compression operator, for which purpose we evaluate the magnitude $\|Ax - Ay\|$; x, y, $A \in S(\vartheta, R)$, i.e.,

$$\|Ax-Ay\|\equiv\|A_1x-A_1y\|$$

$$\overset{\text{def}}{=}\left\|\int_0^t \exp\{-\alpha|\mathbf{v}|^2(t-\tau)\}\left[\int K_e(x'\underset{*}{x'}-x\underset{*}{x})d\omega d\underset{*}{\mathbf{v}}\right]_{t-\tau}d\tau-\int_0^t \exp\{-\alpha|\mathbf{v}|^2(t-\tau)\}\left[\int K_e(y'\underset{*}{y'}-y\underset{*}{y})d\omega d\underset{*}{\mathbf{v}}\right]_{t-\tau}d\tau\right\|$$

$$\leqslant\|(x'\underset{*}{x'}-y'\underset{*}{y'})-(x\underset{*}{x}-y\underset{*}{y})\|\cdot\|L(t,\ \mathbf{v})\|\leqslant(\|x'\underset{*}{x'}-y'\underset{*}{y'}\|+\|x\underset{*}{x}-y\underset{*}{y}\|)\|L(t,\ \mathbf{v})\|$$

$$\leqslant(\|(x'-y')\underset{*}{x'}\|+\|(\underset{*}{x'}-\underset{*}{y'})y'\|+\|(x-y)\underset{*}{x}\|+\|(\underset{*}{x}-\underset{*}{y})y\|)\|L(t,\mathbf{v})\|\leqslant 4R\|L(t,\ \mathbf{v})\|\cdot\|x-y\|.$$

Assuming

$$L_A\overset{\text{def}}{=}4R\|L(t,\ \mathbf{v})\| \tag{21}$$

and using property (18) of the function L(t, $\mathbf{v}$), for any R and a sufficiently small χ_1 we will have

$$\|Ax-Ay\|\leqslant L_A\|x-y\|,\quad L_A<1. \tag{22}$$

The operator A, determined on any closed sphere of the Banach space C under known conditions is thus a compression operator. It transforms $S(\vartheta, R)$ into $S(\vartheta, R)$, provided that

$$\|Ax\| \leqslant \|A_1 x\| + \|F_0\| \leqslant L_A R + \|F_0\| \leqslant R, \quad x \in S(\vartheta, R).$$

From this we have the condition for F_0:

$$\|F_0\| \leqslant (1 - L_A) R. \tag{23}$$

In accordance with the principle of compressed mappings the operator A in $S(\vartheta, R)$ then has a single non-moving point, while Eq. (14) has a unique solution to which the sequence $\{x_{n+1} = Ax_n\}$ converges uniformly for any initial element $x_0 \in S(\vartheta, R)$. The speed of its convergence is characterized by the inequality

$$\|x - x_n\| \leqslant L_A^n (1 - L_A)^{-1} \|x_1 - x_0\|, \tag{24}$$

where x is the exact solution.

The nonnegative solution of the Cauchy problem for (12) can be found by turning to an integral equation in Enskog form, assuming that $F_0 \geq 0$

$$F = \int_0^t \exp\{-\alpha |\mathbf{v}|^2 (t-\tau) - \int_\tau^t [LF]_{t-s} ds\} [NF]_{t-\tau} d\tau + [F_0]_t \exp\left\{-\alpha |\mathbf{v}|^2 t - \int_0^t [LF]_{t-s} ds\right\}. \tag{25}$$

By definition

$$LF \stackrel{\text{def}}{=} \int K_e F d\omega d\underset{*}{\mathbf{v}},$$

$$NF \stackrel{\text{def}}{=} \int K_e F' F' d\omega d\underset{*}{\mathbf{v}}.$$

In operator notation (25) assumes the form

$$F = A_E F, \quad A_E \stackrel{\text{def}}{=} A_{E1} + A_{E2}. \tag{26}$$

The operator A_E is positive and acts in the cone C_+ of nonnegative functions of the space C.

According to the property of the norm, $\|A_E x - A_E y\| \leq \|A_{E1} x - A_{E1} y\| + \|A_{E2} x - A_{E2} y\|$. Since L is a positive operator, with $x, y \in C_+ \cap S(\vartheta, R)$ and $t \geq t_0 \geq 0$ the following relationship is satisfied:

$$\left|\exp\left\{-\int_{t_0}^t [Lx]_{t-s} ds\right\} - \exp\left\{-\int_{t_0}^t [Ly]_{t-s} ds\right\}\right| \leqslant \int_{t_0}^t [L|x-y|]_{t-s} ds. \tag{27}$$

Hence it follows that

$$\|A_{E1} x - A_{E1} y\|$$

$$\stackrel{\text{def}}{=} \left\| \int_0^t \exp\left\{-\alpha |\mathbf{v}|^2 (t-\tau) - \int_\tau^t [Lx]_{t-s} ds\right\} [Nx]_{t-\tau} d\tau - \int_0^t \exp\left\{-\alpha |\mathbf{v}|^2 (t-\tau) - \int_\tau^t [Ly]_{t-s} ds\right\} [Ny]_{t-\tau} d\tau \right\|$$

$$\leqslant \left\| \int_0^t \exp\left\{-\alpha |\mathbf{v}|^2 (t-\tau)\right\} [|Nx - Ny|]_{t-\tau} d\tau \right\| + \left\| \int_0^t \exp\left\{-\alpha |\mathbf{v}|^2 (t-\tau)\right\} \left(\int_\tau^t [L|x-y|]_{t-s} ds\right) [Ny]_{t-\tau} d\tau \right\|, \tag{28}$$

$$\|A_{E2} x - A_{E2}\| \stackrel{\text{def}}{=} \left\| [F_0]_t \left(\exp\left\{-\alpha |\mathbf{v}|^2 t - \int_0^t [Lx]_{t-s} ds\right\} - \exp\left\{-\alpha |\mathbf{v}|^2 t - \int_0^t [Ly]_{t-s} ds\right\}\right\|$$

$$\leqslant \|F_0\| \cdot \left\| \exp\{-\alpha |\mathbf{v}|^2 t\} \int_0^t [L|x-y|_{t-s} ds \right\|. \tag{29}$$

The obvious inequalities

$$\int_{t_0}^t [L|x-y|]_{t-s} ds \leqslant \|x-y\| \int_{t_0}^t [\int K_e d\omega d\underset{*}{\mathbf{v}}]_{t-s} ds, \tag{30}$$

$$|Nx - Ny| \leqslant \|x-y\| 2R \int K_e d\omega d\underset{*}{\mathbf{v}} \tag{31}$$

permit us to contend that

$$\|A_E x - A_E y\| \leqslant (\|L_1(t, \mathbf{v})\| + \|L_2(t, \mathbf{v})\|)\|x - y\| \overset{\text{def}}{=} L_E\|x - y\|, \tag{32}$$

where $L_1(t, \mathbf{v})$ and $L_2(t, \mathbf{v})$ are functions whose explicit form follows (30) and (31); they both diminish as $t \to 0$. Therefore, with a sufficiently small χ_1, the operator A_E is compressive. For F_0 it is easy to find the condition under which $A_E C_+ \cap S(\vartheta, R) \subset C_+ \cap S(\vartheta, R)$. Equation (25) then has a single solution in $C_+ \cap S(\vartheta, R)$, on which the sequence $\{x_{n+1} = A_E x_n\}$ converges uniformly.

It is thus proved that the solutions of the integral equations (14) and (25) can be found as the limits of uniformly converging sequences of a specific form. At the instant $t = \chi_1$, specifying a new initial condition $f_1 \overset{\text{def}}{=} F(\chi_1, \mathbf{r}, \mathbf{v})\exp\{-\alpha(\chi - \chi_1)|\mathbf{v}|^2\}$ and again applying the outlined considerations, we find a solution in the time interval (χ_1, χ_2), then in (χ_2, χ_3), etc., continuing in this manner to any interval at which the solution is limited and diminishes rather rapidly as $|\mathbf{v}| \to \infty$.

Remark. In analogous fashion we can examine the equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \frac{\mathbf{X}(\mathbf{r})}{m}\frac{\partial}{\partial \mathbf{v}}\right)f = \int K(f'f'_* - ff_*)\,d\omega d\mathbf{v}_*, \tag{33}$$

if $\mathbf{X}(\mathbf{r}) = -\text{grad}\, U(\mathbf{r})$ and $U(\mathbf{r}) > -\infty$, i.e., the external field is conservative and shows no infinitely deep potential wells. We then present the solution in the form

$$f(t, \mathbf{r}, \mathbf{v}) \overset{\text{def}}{=} F(t, \mathbf{r}, \mathbf{v})\exp\{-\beta(\chi - t)\varepsilon(\mathbf{r}, \mathbf{v})\},$$
$$\varepsilon(\mathbf{r}, \mathbf{v}) \overset{\text{def}}{=} \frac{1}{2}m|\mathbf{v}|^2 + U(\mathbf{r}), \tag{34}$$

where β is some numerical parameter. In the integral equation corresponding to (14) instead of $\exp\{-\alpha|\mathbf{v}|^2(t - \tau)\}$ we have the term

$$\exp\left\{-\beta\int_\tau^t [\varepsilon(\mathbf{r}, \mathbf{v})]_{t-s}\,ds\right\},$$

because the shift operator $[\ ]_{t-s}$ with respect to the characteristics of the equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \frac{\mathbf{X}(\mathbf{r})}{m}\frac{\partial}{\partial \mathbf{v}}\right)u = 0$$

operates on $\mathbf{r}$ and $\mathbf{v}$. However, $\varepsilon(\mathbf{r}, \mathbf{v})$ is the total energy of the particle moving in the potential field; consequently,

$$[\varepsilon(\mathbf{r}, \mathbf{v})]_{t-s} \equiv \varepsilon(\mathbf{r}, \mathbf{v}).$$

Therefore, for F we obtain the equation

$$F = \int_0^t \exp\{-\beta\varepsilon(\mathbf{r}, \mathbf{v})(t - \tau)\}\left[\int K_e(F'F'_* - FF_*)\,d\omega d\mathbf{v}_*\right]_{t-\tau} d\tau + [F_0]_t \exp\{-\beta\varepsilon(\mathbf{r}, \mathbf{v})t\},$$
$$K_e \overset{\text{def}}{=} K\exp\{-\beta(\chi - t)\varepsilon(\mathbf{r}, \mathbf{v}_*)\}. \tag{35}$$

The nature of the proof subsequently is virtually no different from the case in which $\mathbf{X}(\mathbf{r}) \equiv 0$.

NOTATION

t is the time;
$\mathbf{r}$ is the radius vector of the particle;
m is its mass;
$\mathbf{v}$ is its velocity;
$\mathbf{X}$ is the external field;
σ is the differential scattering cross section;
S is the complete scattering cross section;
ϑ, φ are the scattering angles;

Kn	is the Knudsen number;
f, f_*, f', f'_*	are distribution functions;
$\varphi(r_{ij})$	is the potential of the molecular field;
r_{ij}	is the distance between particles with the numbers i and j;
r_0	is the "clipping" parameter of the potential $\varphi(r_{ij})$;
I	is the collision operator;
K	is its kernel;
$K(\vartheta)$	is the angular portion of the kernel K;
d	is the diameter of the molecule in the model of the solid spheres;
$\alpha, \beta, \chi, \chi_1, \chi_2, \chi_3$	are numerical parameters;
F	is the new unknown function;
F_0	is a function related to the initial condition;
K_e	is the transformed kernel of the collision operator;
$[\ ]_s$	is the shift operator (by time s) with respect to the characteristics of the differential equations;
C	is the space bounded by continuous functions;
C_+	is its cone of nonnegative functions;
$\Vert x \Vert$	is the norm of the element x;
$A, A_1, A_E, A_{E1}, A_{E2}$	are the integral operators effective in C;
$L(t, \mathbf{v}), L_1(t, \mathbf{v}), L_2(t, \mathbf{v})$	are auxiliary functions;
I_1, I_2	are the integrals in (20);
Γ	is the γ-function;
$S(\vartheta, R)$	is a closed sphere in the space C;
$\vartheta \overset{\text{def}}{=} (t, \mathbf{r}, \mathbf{v})$	is the center of this sphere (the null of the space C).
R	is its radius;
L_A, L_E	are the constants of the Lipschitz operators A and A_E;
$U(\mathbf{r})$	is the potential of the external field;
$\varepsilon(\mathbf{r}, \mathbf{v})$	is the total energy of the particles in the field $\mathbf{X}(\mathbf{r})$;
$\overset{\text{def}}{=}$	is the equality sign introduced by definition;
L, N	are operators in A_E.

LITERATURE CITED

1. L. Boltzmann, Lectures on the Theory of Gases [Russian translation], GITTL (1956).
2. M. N. Kogan, The Dynamics of a Rarefied Gas [in Russian], Nauka (1967).
3. T. Karleman, Mathematical Problems in the Kinetic Theory of Gases [Russian translation], IL (1960).
4. E. Wild, Proc. Cambr. Phys. Soc., 47, 602 (1951).
5. D. Morgenstern, Proc. Nat. Acad. Sci., 40, 719 (1954).
6. D. Morgenstern, Journ. of Rat. Mech. and Anal., 4, 533 (1955).
7. H. Grad. Handbuch der Physik, 12, 205 (1958).
8. A. Ya. Povzner, Matem. Sb., 58, No. 1, 65 (1962).
9. B. V. Filippov and G. B. Il'inskaya, Dokl. Akad. Nauk SSSR, 164, No. 1, 768 (1965).